MATH 245 F23, Exam 1 Solutions

1. Carefully define the following terms: composite, converse.

Let n be an integer with $n \geq 2$. We call n composite if there exists some integer m satisfying both $m|n$ and $1 < m < n$. For arbitrary propositions p, q, the converse of conditional proposition $p \to q$ is the proposition $q \to p$.

2. Carefully state the following theorems: Division Algorithm Theorem, Disjunctive Syllogism Theorem.

The Division Algorithm Theorem says: For any integers a, b with $b \geq 1$, there are unique integers q, r satisfying $a = bq + r$ and $0 \leq r < b$. The Disjunctive Syllogism Theorem states: for any propositions p, q, if $p \vee q$ is T and q is F, then p must be T.

3. Use a truth table to help prove the following:
For all propositions p, q, we have $(p \uparrow q), (p \to q) \vdash \neg p$.

Let p, q be arbitrary propositions. Suppose that $(p \uparrow q), (p \to q)$ are both T. Consider the truth table at right. Because $p \uparrow q$ is T, the first row is impossible. Since $p \to q$ is T, the second row is impossible. In both remaining rows, $\neg p$ is T.

p	q	$p \uparrow q$	$p \to q$	$\neg p$
T	T	F	T	F
T	F	T	F	F
F	T	T	T	T
F	F	T	T	T

4. Let p, q be propositions. Without using truth tables, prove $p \wedge q \equiv q \wedge p$.
Note: do not use/cite commutativity of $\wedge$ – you are being asked to prove commutativity!

There are four cases, but three of them end up collapsing. If p, q are both T, then $p \wedge q$ is T, but also $q \wedge p$ is T since q, p are both T. If p, q are not both T (the three cases of p, q both F, or p is F and q is T, or p is T and q is F), then $p \wedge q$ is F, but also $q \wedge p$ is F since q, p are not both T. In all cases, $p \wedge q$ agrees with $q \wedge p$.

ALTERNATE SOLUTION: Suppose $p \wedge q$ is T. By simplification twice, p and q are T. By conjunction, $q \wedge p$ is T. This proves $p \wedge q \vdash q \wedge p$. Now suppose $q \wedge p$ is T. By simplification twice, q and p are T. By conjunction, $p \wedge q$ is T. This proves $q \wedge p \vdash p \wedge q$. Together these prove $p \wedge q \equiv q \wedge p$.

5. Let p, q be propositions. Use semantic theorems to prove the "Trivial Proof Theorem": $q \vdash p \to q$. Do not use the theorem to prove itself!

We begin by assuming q. By addition, $q \vee (\neg p)$. By conditional interpretation, $p \to q$.

6. Prove or disprove: For all $a, b, c \in \mathbb{Z}$, if $ac|b$ then $a|b$.

The statement is true. We begin by letting $a, b, c \in \mathbb{Z}$ be arbitrary, and apply a direct proof. Suppose that $ac|b$. Then there is some integer k with $ack = b$. Set $m = ck$, which is an integer since c, k are. We have $am = b$ for an integer m, so $a|b$.

7. Prove or disprove: For all $x \in \mathbb{Z}$, $|4x + 9| > 1$.

The statement is false, so we need a counterexample. Take $x^\star = -2$. We have $|4x^\star + 9| = |4(-2) + 9| = |1| = 1 \not> 1$.

8. Let $a, b \in \mathbb{N}_0$. Use the definition of $\leq$ to prove that if $2a \leq b$ then $2a^2 + ab \leq b^2$.

We will use a direct proof. Suppose that $2a \leq b$. Then $b - 2a \in \mathbb{N}_0$. Also $b + a \in \mathbb{N}_0$, since $a, b \in \mathbb{N}_0$. But now the product $(b - 2a)(b + a) \in \mathbb{N}_0$, i.e. $b^2 - ab - 2a^2 \in \mathbb{N}_0$. Hence $b^2 - (2a^2 + ab) \in \mathbb{N}_0$, so $2a^2 + ab \leq b^2$.

Note: How would someone come up with the strange idea that $a + b \in \mathbb{N}_0$, and we can multiply by it? Work backwards from the end, and factor $b^2 - 2a^2 - ab$ (which is easy to do if we know that $b - 2a$ is a factor).

9. Prove or disprove: For all $a, b, c \in \mathbb{Z}$, if $a|b$, then $ac^2|b^2c$.

The statement is false, so we need a counterexample. To disprove an implication $p \to q$ we need to make p true and q false, i.e. we need our example to satisfy $a|b$ and $ac^2 \nmid b^2c$. Many counterexamples are possible.

One choice is $a = 2, b = 10, c = 3$. We have $2|10$ since $2 \cdot 5 = 10$. This proves that $a|b$. Now, $ac^2 = 18$ and $b^2c = 300$. If $18k = 300$, then $k = \frac{300}{18} = \frac{50}{3} = 16\frac{2}{3}$, which is not an integer. Hence $ac^2 \nmid b^2c$.

Perhaps the simplest choice is $a = b = 1, c = 2$, although some students don't like simple choices like this. We have $1|1$ since $1 \cdot 1 = 1$. This proves that $a|b$. Now, $ac^2 = 4$ and $b^2c = 2$. If $4k = 2$, then $k = 0.5 \notin \mathbb{Z}$. Hence $ac^2 \nmid b^c$.

10. Prove or disprove: $\forall x, y \in \mathbb{R}$, $(x < y) \to (\exists z \in \mathbb{R}, x < z < y)$.

The statement is true. We begin by letting $x, y \in \mathbb{R}$ be arbitrary. Via direct proof, we assume that $x < y$.

We now need to prove $\exists z \in \mathbb{R}, x < z < y$; it's a little tricky to find such a z. The usual method is to take the midpoint, i.e. take $z = \frac{x+y}{2} = \frac{x}{2} + \frac{y}{2}$. Now since $x < y$ we have $\frac{x}{2} < \frac{y}{2}$. Adding $\frac{x}{2}$ to both sides gives $x < z$, while adding $\frac{y}{2}$ to both sides gives $z < y$. Combining these, we get $x < z < y$.